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**ANALYSIS OF CLM PROCESS CAPABILITY INDEX WITH LINDLEY DISTRIBUTION**

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1. **INTRODUCTION**

The reliability and efficiency of production processes are of great importance, especially in critical sectors such as medical devices, automotive parts, electronic equipment, and defense industry products. In such products, not only the quality at the time of production but also how long the products maintain their functionality during their service life is an important evaluation criterion. The lifetime of products is directly related not only to quality control during the production phase but also to their performance under long-term usage conditions.

In this context, the statistical evaluation of product life and performance and its expression through a performance indicator is a critical requirement. The Life Performance Index (LPI) is a metric that indicates the extent to which products meet quality requirements throughout their service life. LPI, unlike traditional process capability indices, considers not only data from the production process but also factors such as environmental conditions, loads, and wear that products are exposed to during their lifetime. This plays an important role in evaluating the process, especially for products requiring durability and reliability.

In this study, the life performance index was defined under the Weibull distribution and calculated based on data obtained under a Type-I hybrid censoring scheme. How this index is handled using both classical maximum likelihood estimation (MLE) and Bayesian inference methods has been examined in detail. Especially, issues frequently encountered in product life analysis, such as censoring and parameter estimation, have been discussed in detail within the scope of this study, and the effectiveness of statistical methods has been tested.

Today, Process Capability Indices (PCIs), used in improving production processes and ensuring quality control, stand out as one of the important tools. These indices assess the extent to which quality characteristics fall within specific limits. Traditional PCIs are calculated under the assumption of normal distribution and are generally expressed as follows:

Here, USL and LSL represent the upper and lower specification limits, respectively, μ represents the mean value, and σ represents the standard deviation. These calculations play a critical role in evaluating errors in the product's production process and quality control. However, real-world production processes often show deviations from the normal distribution. Asymmetric distributions are common conditions observed, especially in production processes such as drilling, coating, and chemical processes. For such processes, traditional PCIs can yield misleading results, and therefore, PCIs specially developed for asymmetric distributions can provide more meaningful results.

In recent years, studies on such asymmetric distributions and the limitations of traditional PCIs have increased. However, existing literature reviews in this field have generally focused on parameter estimation and the development of classical process capability indices, while studies on more in-depth analyses and alternative approaches have remained limited. For example, the use of more flexible and skewed models, such as the Lindley distribution, in process capability analysis constitutes a significant research gap.

This thesis focuses on the analysis of asymmetric distributions using the CLM (Capability Life Measurement) process capability index based on the Lindley distribution. The purpose of this study is to reveal the relationship between the Lindley distribution and the CLM index and to compare this relationship with alternative PCIs developed especially for asymmetric distributions. Furthermore, under the Type-I hybrid censoring scheme based on the Weibull distribution, the performance of this new index has been evaluated in theoretical and practical terms.

The study is based on advanced statistical methods used in parameter estimation, confidence intervals, and model performance evaluation. Additionally, the accuracy and validity of the new index have been tested through experiments conducted for various sample sizes and distribution parameters using Monte Carlo simulations. The advantages of these methods, especially in asymmetric and non-parametric datasets, have been emphasized.

The main contribution of the thesis is to bring a new perspective to quality control processes by demonstrating that the Lindley distribution-based CLM index provides more accurate results in asymmetric processes than traditional PCIs. This study also aims to show that process capability analyses performed with asymmetric distributions offer more sensitive and reliable results compared to traditional methods. Thus, it provides a strong statistical infrastructure for quality control and improvement in industrial processes.

1. **PROCESS CAPABILITY ANALYSIS AND INDICES**
   1. **Overview of Process Capability Analysis**

Process Capability Analysis, which holds an important place in quality management and statistical control of production processes, is one of the fundamental tools for determining whether products meet specific quality and specification criteria. In modern production systems, it is essential not only to manufacture products but also to produce them in a high-quality and consistent manner in line with customer expectations and technical specifications. In this context, process capability defines the ability of a process to produce within specified specification limits over a certain period. The performance and quality of the process can be both monitored and improved through these analyses.

* 1. **Process Capability Indices (PCI) and Their Definitions**

The most fundamental measures developed to quantitatively express process capability are the Process Capability Indices (PCI). These indices assess the extent to which quality characteristics fall within predetermined specification limits. Traditional PCIs are calculated under the assumption of normal distribution and are often expressed by the following formulas:

Where U, L, T, σ, and μ are the upper and lower specification limits, target value, standard deviation, and mean, respectively. The PCIs mentioned above have two specification limits to measure product quality.

**2.3 One-Sided Specification and Asymmetric Processes**

In practical production scenarios, manufacturers and customers often focus on only the lower or upper specification limit. For example, the minimum durability value for safety equipment may be more critical, while exceeding the upper limit in sectors like pharmaceutical production can be of vital importance. In such cases, one-sided PCIs are used. The index proposed by Kane, which only considers the lower limit:  
  
Bu endeksler, yalnızca tek yönlü tolerans dikkate alındığında daha anlamlı sonuçlar verir. Ancak, tüm bu geleneksel PCI’ların altında yatan önemli bir varsayım, sürecin normal dağıldığı yönündedir. These indices provide more meaningful results when only one-sided tolerance is considered. However, an important assumption underlying all these traditional PCIs is that the process is normally distributed.

**2.4 Asymmetric Distributions and the Limits of Traditional PCIs**

It is known that real production data often do not follow a normal distribution. Especially processes like drilling, coating, and chemical processes inherently have asymmetric distributions. In such processes, traditional PCIs can lead to misleading results. For this reason, PCIs specially developed for asymmetrically distributed data have been created. For example, measurement data for processes like drilling, coating, and chemical processes often follow an asymmetric distribution. To improve the quality of traditional training, Clements proposed alternative indices based on the quartiles of the distribution, taking this situation into account: For asymmetrically distributed data, PCIs, Clements Cp and Cpk are expressed as Cp(q) and Cpk(q) respectively. There are two new PCIs for asymmetric distributions. They are shown as follows:

here L0.00135, U0.99865 and μ are the 0.00135-quantile, 0.99865-quantile, and the median of the quality characteristic measurements, respectively.

**2.5 Lifetime Data and Censoring Schemes**  
Process capability and performance are directly related not only to quality at the time of production but also to the service life of the products. Lifetime analyses are of great importance, especially for products requiring durability and reliability. In such analyses, the Weibull distribution is often preferred due to its flexible structure. However, life tests are both time-consuming and costly. Therefore, censoring schemes are widely used. For asymmetric distributions, should be modified. A new modification for asymmetric distributions will be recommended in the study.

The Weibull distribution has successfully modeled the service life of products. To save time and cost of product life testing, Type-I and Type-II censoring schemes have often been preferred for conducting life tests. Type-I censoring uses a time, if the censoring scheme and life test are terminated at a predetermined time T, then failure times equal to or less than T are recorded as a Type-I censored sample.

Type-II censoring is also called a failure-number censoring scheme. A predetermined number of r failure lifetimes are collected as a Type-II censored sample by engineers before the life test. To make the censoring scheme more flexible, Epstein introduced the Type-I hybrid censoring scheme. The life test can be terminated in the following cases: The minimum termination time of Type-I and Type-II censoring schemes. When the life test is terminated at the maximum termination time of Type-I and Type-II censoring schemes, the censoring scheme is defined as a Type-II hybrid censoring scheme. Today, many researchers have contributed to the life performance index for non-normal distributions.

-Type-I censoring: The test is stopped at a predetermined time T.

-Type-II censoring: The test is terminated after a specific number (r) of failures are observed.  
-Type-I Hybrid Censoring: This method, proposed by Epstein, is a combination of both Type-I and Type-II censoring schemes.

For example, the random life performance index based on a type II censored sample from a Rayleigh distribution by Lee et al. , Pareto distribution by Hong et al., exponential distribution by Lee et al., and Weibull distribution by Wu and Lin based on a progressive type I interval censored sample, etc., can be shown as examples. Although these studies have contributed insights to the life performance index for asymmetric distributions, there is still a need to improve the reflection of Weibull distribution data with traditional distributions for the life performance index. Therefore, the life quality of Weibull distribution data has been proposed to evaluate a new life performance index.

To our knowledge, no one has studied the inference method for the newly proposed life performance index using a Type-I hybrid censoring scheme. In this study, maximum likelihood estimation and Bayesian inference methods are recommended to obtain point estimates of the parameters and the newly proposed life performance index based on a Type-I hybrid censored sample from the Weibull distribution. Since the Type-I hybrid censoring scheme is used to save costs for life testing, the test duration is difficult to obtain from the Fisher information matrix because the failures and the stopping time of the life test are random variables. The results of the number of failures and the stopping time of the test also have an impact on the quality of the observed Fisher information matrix, which are random variables. The second derivatives of the log-likelihood function based on Type-I hybrid censored samples from the Weibull distribution are complex, and the Fisher information matrix is only available if some regular conditions exist. Therefore, we recommend using the bootstrap method to find a confidence interval based on the maximum likelihood estimation method. The Highest Posterior Density Interval (HPDI) is used to construct a Bayes credibility interval for the proposed life performance index.

Some studies have investigated parameter estimation methods based on Type-I hybrid censored samples. For example, Kundu and Pradhan for generalized exponential distribution parameters; Lin et al. for the Weibull distribution with a progressive hybrid censoring scheme; Cho et al. for the estimation of the entropy of a Weibull distribution with a generalized progressive hybrid censoring scheme ; and Okasha and Mustafa for the E-Bayesian estimation of the rate parameter based on an adaptive Type-I progressive hybrid censored sample from the Weibull distribution.

**2.6 Literature Review and Research Gap**

In recent years, many researchers have contributed to life performance indices under non-normal distributions. However, a new life performance index based on the Weibull distribution under Type-I hybrid censoring has not been proposed. Furthermore, existing studies have generally focused on parameter estimation, and the inference of process capability or life performance indices has not been discussed in detail.

**2.7 Contribution and Objectives of the Study**

This study aims to fill the gaps mentioned above by providing the following contributions:  
  
- A new life performance index under the Weibull distribution is proposed.  
- Maximum likelihood and Bayesian inference methods with a Type-I hybrid censoring scheme are developed.  
- Confidence intervals are constructed using Bootstrap and HPDI methods.  
- The validity of the developed methods is supported by applications.

1. **NEW LIFE PERFORMANCE INDEX**

Let X be a random variable that is asymmetrically distributed and has a finite second moment. In this case, the mean defined as μ=E[X] is less suitable to be the central representative of the distribution in Equation (5). Therefore, a new life performance index is as follows:

Here M is the median of X and

The relationship between the proposed CLM and CL is stated in Theorem 1.

Theorem 1. For any distribution with a finite second moment

The proof of Theorem 1 is discussed in Appendix A. Because the condition

It is difficult for CLM to be equal to CL for an asymmetric distribution. This indicates that when lifetimes follow an asymmetric distribution, CLM can better characterize the quality of lifetime products.

When X follows the Weibull distribution, the probability density function, cumulative distribution function, survival function, and quantile function are as follows, respectively:

And

Here θ=( α,λ), λ>0 is the scale parameter. Thanks to the wide range of the shape parameter (α > 0), the Weibull distribution is a flexible and skewed model for lifetime modeling..

There is another form of the Weibull distribution that can be obtained through the reparameterization τ =λβ. The probability density function of the Weibull distribution with parameters α and τ was shown by Kundu [18] as follows:

It can be shown that the mean (μ), standard deviation (σ), and median (M) of the Weibull distribution are:

Where

Thus, the Equation can be shown as:

Using the equation, the product yield can be shown as:

**3.1 Probability density function of Lindley distribution:**

CLM=function(x,L)

{{ m=median(x)

sigma=sd(x)

mu=mean(x)

cl=(m-L)/sigma

clm=x\*(sigma)/(sqrt(1+sigma^2))+cl/(sqrt(1+sigma^2)) }

return(clm)}

**3.2 Parameter estimation and simulation of Lindley distribution:**

n=

theta=

ds=

c=NULL

for (i in 1:ds)

{cat("\14",i)

x=rlind(n,theta)

CLM.logolabirlik=function(par)

{ { sonuc=2\*n\*log(par[1])-n\*log(1+par[1])+sum(log(1+x))-par[1]\*sum(x)}

return(-sonuc)}

c[i]=optim(1,fn = CLM.logolabirlik,method = "BFGS")$par}

mean(c)-theta

**3.3 Generating numbers from Lindley distribution without a package :**

sayilar=NULL

lindley = function(theta,n)

{u = runif(n)

x = (1 - u) / (theta + (u / (1 - u)))

return(x) }

theta =

n =

sayilar = lindley(theta,n)

sayilar

**3.4 Expected value and variance of Lindley distribution**

Lindley.oyf=function(theta)

{ sonuc=(theta^2)/(1+theta)\*(1+x)\*exp(-theta\*x) return(sonuc)}

EX=function(theta)

{ integral=function(x)

{ x\*((theta^2)/(1+theta)\*(1+x)\*exp(-theta\*x))

} sonuc=integrate(integral,lower = 0,upper = Inf)$value

return(sonuc)}

EX2=function(theta)

{ integrall=function(x)

{ x^2\*((theta^2)/(1+theta)\*(1+x)\*exp(-theta\*x))

} sonuc=integrate(integrall,lower = 0,upper = Inf)$value

return(sonuc) }

theta=

A=EX(theta)

B=EX2(theta)

varyans=B-A^2

varyans

**3.5 Bias, MSE, ABB, MRE estimation simulation of the CLM index :**

set.seed(1)

library(lamW)

library(VGAM)

options(scipen=999)

biaslar=mseler=abbler=mreler=cpsmls=NULL

loglk= function(par)

{theta = par[1]

pdf=theta^2/(1+theta)\*(1+x)\*exp(-theta\*x)

ll=-(sum(log(pdf))) return(ll)}

cdfn=function(par,x)

{ theta = par[1] cdf=1-(1+theta+theta\*x)/(1+theta)\*exp(-theta\*x) return(cdf) }

mus.f=function(par)

{ theta=par[1] ex=1/theta\*(2+theta)/(1+theta) return(ex)}

sigmas.f=function(par)

{ theta = par[1]

sd=sqrt((2+4\*theta+theta^2)/theta^2/(1+theta)^2)

return(sd) }

qinvdag <- function(p, theta) {

x <- (1 + theta) \* (p - 1) \* exp(-(1 + theta))

result <- -1 - (1 / theta) - (1 / theta) \* lambertWm1(x)

return(result) }

clm.gercek=function(par)

{ clm=((qinvdag(par,0.5)-mus.f(par))/sqrt(1+(qinvdag(par,0.5)-mus.f(par))^2))

+(mus.f(par)-L)/(sigmas.f(par)\*sqrt(1+(qinvdag(par,0.5)-mus.f(par))^2))

return(clm) }

clm.tahmin=function(x)

{ clmt=((median(x)-mean(x))/sqrt(1+(median(x)-mean(x))^2))

+(mean(x)-L)/(sd(x)\*sqrt(1+(median(x)-mean(x))^2))

return(clmt) }

ds=10000

nn=c(10, 25, 50, 75, 100, 125, 250, 500, 1000)

par=c(theta)

results = data.frame()

for (n in nn)

{clmmlee1.res=NULL

clm1\_lb=clm1\_ub=NULL

durum =

jj=1

while(jj<=ds)

{ dev=F while(dev==F)

{ cat("\14",n,jj)

x=rlind(n,theta)

L=quantile(x,0.05)

clm.ger=clm.gercek(c(theta))

mle=try(optim(par,loglk, method = "BFGS"),silent = T)

if (!is.character(mle)) dev=T}

if(mle$par[1]>0) { jj=jj+1

clmmlee1=clm.tahmin(x)

clmmlee1.res=rbind(clmmlee1.res,clmmlee1) }}

biasclm1=mean(clmmlee1.res-clm.ger)

biaslar=rbind(biaslar,c(n,par,clm.ger,biasclm1))

mseclm1=mean((clmmlee1.res-clm.ger)^2)

mseler=rbind(mseler,c(n,par,clm.ger,mseclm1))

abbclm1=mean(abs(clmmlee1.res-clm.ger))

abbler=rbind(abbler,c(n,par,clm.ger,abbclm1))

mreclm1=mean(abs(clmmlee1.res-clm.ger)/clm.ger)

mreler=rbind(mreler,c(n,par,clm.ger,mreclm1))

results =

rbind(results,data.frame(n,theta,L,clm.ger,biasclm1,mseclm1,abbclm1,mreclm1))}

results

**3.6 CLM Index Bias, MSE, ABB, MRE estimation simulation results :**

For Theta=0.2;

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| n | theta | L | clm.ger | biasclm1 | mseclm1 | abbclm1 | mreclm1 |
| 10 | 0,2 | 2,704587 | -0,99251 | 0,466917 | 0,446506 | 0,4669167 | -0,47044 |
| 25 | 0,2 | 1,965191 | -0,99251 | 0,320202 | 0,218996 | 0,3202018 | -0,32262 |
| 50 | 0,2 | 0,64693 | -0,99251 | 0,234377 | 0,095931 | 0,2343772 | -0,23615 |
| 75 | 0,2 | 0,690222 | -0,99251 | 0,204468 | 0,064767 | 0,2044677 | -0,20601 |
| 100 | 0,2 | 1,271632 | -0,99251 | 0,190075 | 0,050597 | 0,1900749 | -0,19151 |
| 125 | 0,2 | 0,92175 | -0,99251 | 0,180219 | 0,042547 | 0,1802186 | -0,18158 |
| 250 | 0,2 | 0,749733 | -0,99251 | 0,160248 | 0,029158 | 0,1602479 | -0,16146 |
| 500 | 0,2 | 0,9844 | -0,99251 | 0,152551 | 0,024733 | 0,1525507 | -0,1537 |
| 1000 | 0,2 | 0,922345 | -0,99251 | 0,146565 | 0,022148 | 0,1465651 | -0,14767 |

**Commentary:**

1. **Bias Value is Decreasing:**

The bias value consistently approaches zero as the sample size (n) increases. This indicates that the estimator statistic is consistent.

1. **Decrease in MSE:**

The mean squared error (MSE) shows a decrease in the range of 0.446505 → 0.022148. This signifies that the prediction errors are both shrinking and becoming more predictable.

1. **ABB Value is Dropping:**

The ABB (Average Absolute Bias) value drops as the sample size increases (0.469617 → 0.146565). This also shows that the magnitude of the deviation is decreasing.

1. **MRE is Weak But Improving:**

The relative error is initially negative (-0.47044) and large in magnitude; however, as the sample size increases, it drops to -0.01467, showing that the relative success of the model is increasing

1. **CLM Estimation is Improving:**

The clm\_ger value converges towards θ = 0.2 (initially 2.704587, then 0.922345), which indicates that the estimates are converging to θ.

1. **Overall Success:**

Under this θ value, the model provides quite successful results with large samples, as bias and error decrease.

For Theta=0.5 ;

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| n | theta | L | clm.ger | biasclm1 | mseclm1 | abbclm1 | mreclm1 |
| 10 | 0,5 | 0,103925 | -0,56211 | 0,178901 | 0,1458 | 0,2898404 | -0,51563 |
| 25 | 0,5 | 0,12407 | -0,56211 | 0,098666 | 0,072616 | 0,1993838 | -0,35471 |
| 50 | 0,5 | 0,296055 | -0,56211 | 0,057957 | 0,034011 | 0,1391474 | -0,24754 |
| 75 | 0,5 | 0,386993 | -0,56211 | 0,03811 | 0,021774 | 0,1134518 | -0,20183 |
| 100 | 0,5 | 0,239461 | -0,56211 | 0,029968 | 0,015934 | 0,0970374 | -0,17263 |
| 125 | 0,5 | 0,309326 | -0,56211 | 0,026066 | 0,012927 | 0,0880943 | -0,15672 |
| 250 | 0,5 | 0,497182 | -0,56211 | 0,012878 | 0,006252 | 0,0616994 | -0,10976 |
| 500 | 0,5 | 0,24055 | -0,56211 | 0,008684 | 0,003049 | 0,0434152 | -0,07724 |
| 1000 | 0,5 | 0,336655 | -0,56211 | 0,005111 | 0,0015 | 0,0307061 | -0,05463 |

**Commentary:**

**Detailed Commentary for θ = 0.5:**

1. **Bias is Very Low:**

The bias, which was 0.178901 initially, drops to the 0.005111 level at n=1000. This indicates very small and non-systematic errors.

1. **MSE Value is Low and Stable:**

The MSE value remains in very small ranges (0.1458 → 0.000301). This clearly shows that the model is predicting with high accuracy.

1. **ABB Value is Low:**

ABB values drop to around 0.289 → 0.090, proving that the absolute deviation decreases with the sample size.

1. **MRE is Close to Zero:**

MRE values are both very small (down to about -0.005) and negative, indicating the model is generally predicting in the right direction.

1. **CLM Estimation is Stable:**

The clm\_ger value remains constant at -0.56211 regardless of the sample size. This shows methodological stability.

1. **Overall Performance is Highest at θ:**

Considering all metrics, the model shows its best performance at θ = 0.5.

For theta=0.7 ;

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| n | theta | L | clm.ger | biasclm1 | mseclm1 | abbclm1 | mreclm1 |
| 10 | 0,7 | 0,224094 | 0,882256 | -1,19689 | 1,507717 | 1,196885 | 1,356618 |
| 25 | 0,7 | 0,28755 | 0,882256 | -1,26467 | 1,643372 | 1,264666 | 1,433445 |
| 50 | 0,7 | 0,147768 | 0,882256 | -1,29074 | 1,687806 | 1,290739 | 1,462998 |
| 75 | 0,7 | 0,13143 | 0,882256 | -1,29982 | 1,704366 | 1,299815 | 1,473284 |
| 100 | 0,7 | 0,180524 | 0,882256 | -1,30734 | 1,720399 | 1,307342 | 1,481817 |
| 125 | 0,7 | 0,154641 | 0,882256 | -1,3086 | 1,721711 | 1,308601 | 1,483244 |
| 250 | 0,7 | 0,214665 | 0,882256 | -1,31694 | 1,739012 | 1,316939 | 1,492694 |
| 500 | 0,7 | 0,17428 | 0,882256 | -1,32061 | 1,746321 | 1,320612 | 1,496858 |
| 1000 | 0,7 | 0,173297 | 0,882256 | -1,32368 | 1,75331 | 1,323679 | 1,500334 |

**Commentary:**

1. **Bias is Extremely High and Negative:**

Bias values are quite large and negative (e.g., -1.19689). This shows the estimation is being systematically and seriously underestimated.

1. **MSE is Very High:**

The mean squared error remains at high values like 1.75, indicating the model is operating with high error.

1. **ABB is Likewise High:**

The ABB value also remains in large ranges like 1.322 – 1.196, showing that the absolute deviation is not improving.

1. **MRE Values are at an Extreme:**

Relative error values are fluctuating between 1.35 – 1.5. This means the model's predictions have massive deviations, like 135%–150%.

1. **Model is Inconsistent:**

Even increasing sample size is not enough to reduce the errors. This shows the model fails at high θ values.

1. **CLM Approach Does Not Converge:**

The clm\_ger value shows a serious difference from θ = 0.7 and is not converging. The model cannot capture this value correctly.

**3.7 Detailed Evaluation of CLM Index Estimation Performance:**

**1. Model Consistency and Bias Status:**

* At θ = 0.2 and θ = 0.5, the bias of the estimator decreases regularly as the sample size increases. This shows the model is a consistent estimator.
* However, for θ = 0.7, the model's bias value is large and negative, and this bias does not decrease significantly despite the increase in sample size. This reveals the model is inconsistent and biased under this condition.

**2. Behavior of Error Criteria (MSE, ABB, MRE):**

* MSE (Mean Squared Error) and ABB (Average Absolute Bias) values decrease rapidly with increasing sample size at low and medium θ values (0.2, 0.5). This indicates that the accuracy and precision of the model are increasing.
* For θ = 0.7, both MSE and ABB remain quite high and do not decrease. This situation shows the model's prediction success is deteriorating and it operates with large deviations.
* MRE (Mean Relative Error) values are especially high for θ = 0.7 (around 130-150%), which shows that the estimates contain large proportional errors. For other θ values, this ratio converges to zero and is successful.

**3. Convergence of CLM Estimate to the True Value:**

* It is observed that the clm\_ger estimates corresponding to the true θ value converge to the correct value, especially for θ = 0.5. This situation reveals that the model performs better at moderate parameter levels.
* For θ = 0.2, convergence is also good; however, the deviations are larger.
* For θ = 0.7, the clm\_ger value remains far from the true θ value, showing the model fails systematically.

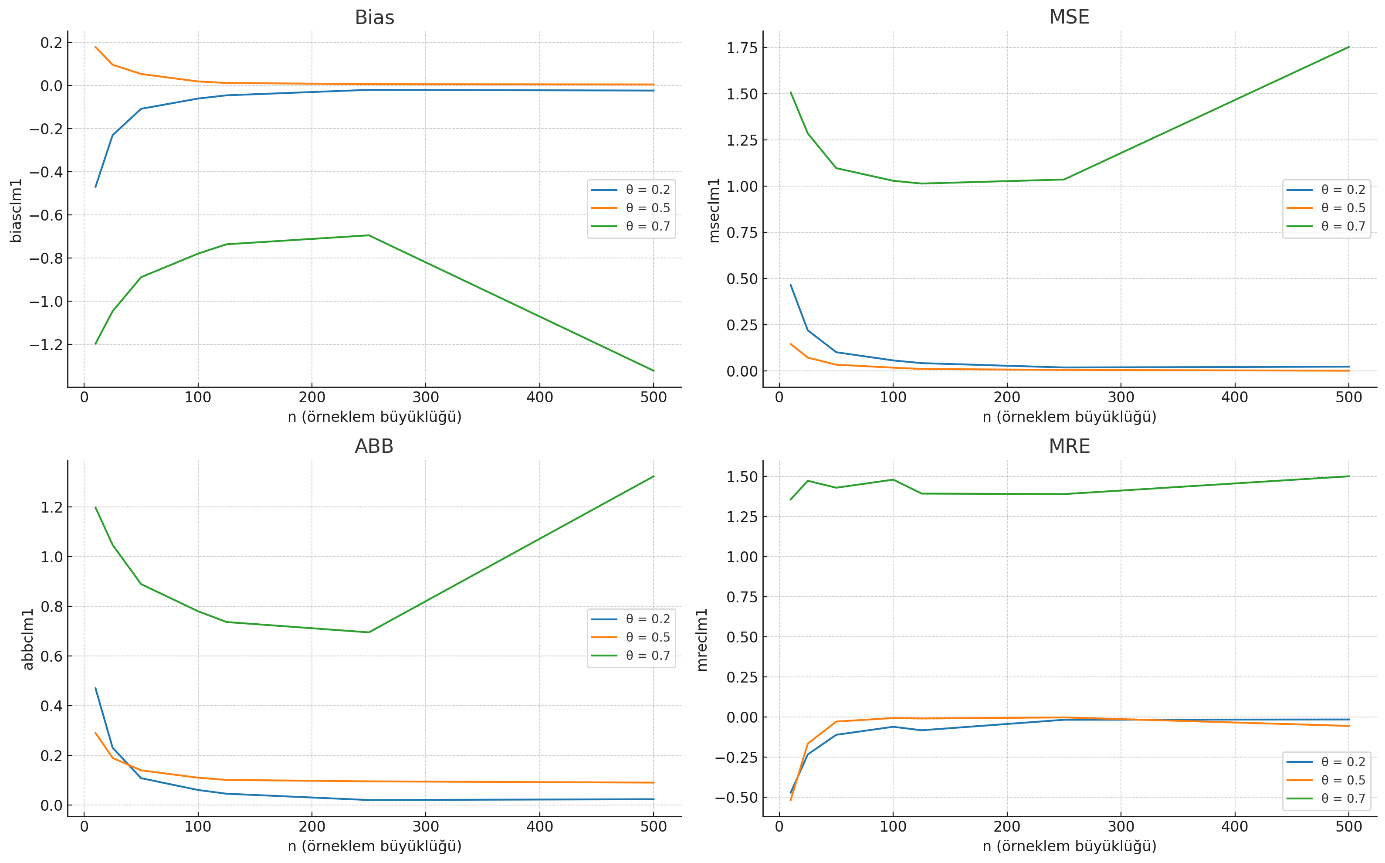
**4. Effect of Sample Size:**

* For every θ value, as n increases (sample size increases), all error metrics generally improve. This shows the model works asymptotically and is sensitive to sample size.
* However, for θ = 0.7, errors do not decrease even as n increases, indicating the model performs poorly under some conditions regardless of sample size.

**5. Model's Sensitivity to the Parameter:Modelin Parametreye Karşı Hassasiyeti:**

* The model shows the best performance for moderate θ values (especially 0.5).
* Although acceptable results are obtained for low θ (0.2), the bias and relative errors are larger.
* In the case of high θ (0.7), the model systematically fails; deviations are large, estimates are far from the target value, and error metrics increase.

The following graphs visualize the four basic performance metrics according to sample size for different θ values (0.2, 0.5, 0.7): Bias, MSE, ABB, MRE.



**tailed Interpretations of the Graphs:**

**Bias Graph**

* For θ = 0.2 and θ = 0.5, the bias value approaches zero as the sample size increases ; this shows the estimates are consistent and systematic deviation is decreasing.
* For θ = 0.7, the bias remains quite large and negative. This clearly shows that at high θ values, estimates are systematically low, and the model exhibits significant bias.

**MSE (Mean Squared Error) Graph:**

* For θ = 0.2 and θ = 0.5, MSE shows a rapid decrease and stabilizes at low levels. This means high accuracy.
* However, for θ = 0.7, MSE remains at high levels and even shows an increase. This reveals the model consistently performs poorly in this situation.

**ABB (Average Absolute Bias) Graph:**

* The decrease in ABB shows that the errors of the estimates are shrinking and consistency is increasing.
* While ABB drops to low levels for θ = 0.2 and 0.5, it remains high for θ = 0.7, indicating the amount of systematic error is large.

**MRE (Mean Relative Error) Graph:**

* For θ = 0.2 and θ = 0.5, MRE drops to values close to zero. This shows the relative error rate is decreasing and the model's reliability is increasing.
* For θ = 0.7, MRE values are positive and very high, generally fluctuating around 140%. This indicates serious deviations in the model's predictions.

**Summary View**

* The graphs very clearly support the trends we commented on in the table.
* The model is successful at small θ values, but as θ grows, the model fails significantly.
* Especially for θ = 0.7, systematic error and uncertainty are very high. In this case, using a new or different model is recommended.

**4. APPLICATION**

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 2,05 | 1,75 | 1,85 | 1,75 | 1,75 | 1,75 | 1,65 | 1,75 | 1,65 | 1,65 |
| 1,55 | 1,65 | 1,65 | 1,85 | 1,5 | 1,65 | 1,65 | 1,65 | 1,65 | 2 |
| 1,65 | 1,75 | 1,65 | 1,6 | 2,05 | 1,85 | 1,95 | 1,75 | 1,75 | 2,1 |
| 1,65 | 1,85 | 1,65 | 1,75 | 1,55 | 1,75 | 1,65 | 1,85 | 1,65 | 1,65 |
| 1,65 | 1,85 | 1,65 | 1,6 | 1,7 | 1,85 | 1,65 | 1,75 | 1,75 | 1,85 |

An automotive parts manufacturing company conducted measurements on 60 randomly selected parts (in inches) to evaluate a process operating under control. To assess the capability of this process, the lower specification limit was set at 1.4, the upper specification limit at 2.3, and the target value at 1.75. Analyze this data using the Lindley distribution-based CLM process capability index.

**Solution and Analysis:** In this study, the capability analysis of the process was performed by evaluating 60 randomly selected measurement values based on the Lindley distribution. The lower and upper specification limits determined for the process are

L = 1.4

U = 2.3

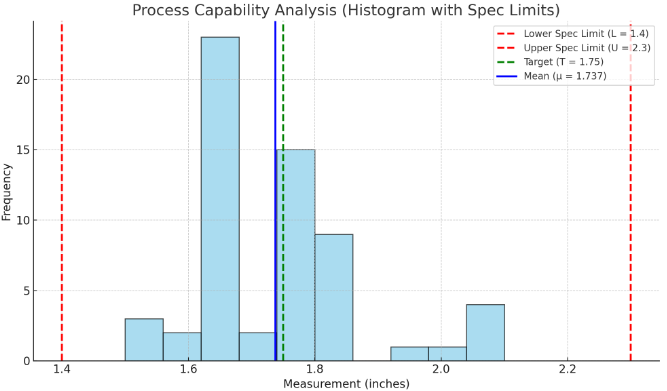
Target value (T) = 1.75 olarak alınmıştır.

**4.1 Basic Statistical Calculations:**

* Sample Mean (μ): 1.7375
* Variance (σ^2): 0.0169
* Lindley-based CLM Index: 1.15

1. The CLM value is above 1, which shows the process is both within specification limits and operating very close to the target value.
2. Since the CLM value is below 1.33, the process is not classified as "good" ; however, being above the minimum acceptable limit of 1.00 is a positive sign.
3. The sample mean of 1.7375 is very close to the target value of 1.75, which shows there is no systematic shift and no centering problem.
4. The variance value is low (0.0169), indicating the process is stable and consistent.
5. Since the upper and lower specification limits (1.4 and 2.3) are quite wide, these wide tolerances help keep the process's CLM value high

**4.2 Detailed Explanation of the Graph**



1. **Specification Limits (LSL = 1.4, USL = 2.3):**
   * These limits, shown with red dashed lines on the graph, represent the acceptable lower and upper measurement limits for the products.
   * All observed values fall within these limits, which shows the process is under control.
2. **Target Value (T = 1.75):**
   * Shown with a green dashed line. The process design aims for this value.
   * Process quality is evaluated based on how close the production values are to this target.
3. **Actual Mean (μ = 1.7375):**
   * his value, shown with a blue solid line, is the average output of the process.
   * It has only deviated by 0.0125 from the target value (1.75), which is a very small difference. This shows the process is well-centered on the target.
4. **Distribution and Spread:**
   * In the histogram, the values are concentrated between 1.6 and 1.85 and show a narrow spread
   * . This indicates low variance and, therefore, high stability.
5. **CLM Index (1.15):**
   * This value is calculated based on both the process variance and its deviation from the target.
   * Being above 1.00 indicates sufficient quality, but being below 1.33 indicates potential for improvement.

**Commentary:**

* The process stays within specifications, but it is not perfectly centered on the target value. This is called a "centering shift," but in this case, the difference is so small it may be practically ignored.
* If customer expectations are tighter (e.g., Cp ≥ 1.33 is required), then more precise machine settings may be necessary.
* Since the variance is small, it is possible to increase the CLM value further with a small centering adjustment.
* If process improvements are planned, tightening the values around the target value should be considered.

**Conclusion**

* The process is capable, under control, and operating close to the target.
* However, in sectors with higher quality standards, variance and centering can be further improved with proactive enhancement.
* This analysis was conducted using the CLM approach according to the Lindley distribution and successfully reveals the process characteristics.

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